A Note on MGR Methods

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ABSTRACT

By MGR we denote a class of special, highly efficient multigrid methods for solving h-discrete elliptic differential equations. Unlike standard multigrid methods [4, 9, 18], MGR methods are characterized by "intermediate" grids (between the given h-grid and the $2h$-grid) and by special fine-to-coarse and coarse-to-fine grid transfer techniques. The MGR idea has been conceived [13, 7] by the second author in trying to extend the range of applicability of the total-reduction method [15, 16] to more general problems. Described in a somewhat different way, methods of MGR type have in the meantime also been considered by Braess [1–3] and Meis [11]. The convergence properties of the simplest MGR method (MGR-0) are essentially improved if it is combined with one step of checkered Gauss-Seidel iteration (MGR-CH). For the model problem of Poisson's equation in the unit square, for example, the spectral radius decreases from $\frac{1}{2}$ to $\frac{2}{27}$ by such a modification, whereas the computational effort is not essentially enlarged. For Poisson-like equations, MGR-CH yields the fastest iterative solver known so far. Other MGR variants are particularly suitable for anisotropic operators.

1. INTRODUCTION

In [7], the development of "nonstandard" multigrid (MG) methods for linear elliptic boundary-value problems (in 2D regions) was outlined. It was pointed out that these methods have several advantages in *practical* respects: Compared to "standard" MG-techniques [4, 9, 10, 18] they are

(1) considerably faster—even on sequential computers,
(2) just as widely applicable to general problems,
(3) immediately suitable for parallel processing (see also [6]).

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Perhaps still more important than these practical advantages is the fact that the nonstandard techniques can be *rigorously analyzed* quantitatively in the following sense: By Fourier expansion the spectral radii (asymptotic convergence factors) and the spectral norms (error-reducing factors) of the corresponding two-level $(h, 2h)$ methods can be precisely evaluated for *model problems*. (From these quantitive results on the two-level methods, the convergence properties of the complete multilevel methods can be analyzed in a well-known straightforward manner [18].)

Examples for such model problems are the discrete Poisson and the Helmholtz equations, as well as the anisotropic equation

$$-au_{xx} - bu_{yy} + cu = f,$$

with Dirichlet, Neumann, or periodic boundary conditions in rectangular 2D regions, using 5- or 9-point discretizations of order 2 or 4, respectively. (Also, 4th-order equations of biharmonic type and 3D equations may be treated.) Each of them stands for a class of more general problems (with variable coefficients, in nonrectangular convex regions, etc.) to which the quantitative results obtained for the corresponding model problem carry over practically. Thus—in contrast to many of the general approaches in the convergence theory of MG methods—the elementary "model problem analysis" allows us to evaluate the different algorithms and to find "optimal" ones.

Among the nonstandard techniques the so-called *MGR methods* are of particular interest. The essential algorithmic features of these methods are the use of intermediate grids (between the given h- and the $2h$-grids) and the specific coarse-to-fine and fine-to-coarse grid transfers. These transfer techniques and the use of intermediate grids have been motivated by the total-reduction (TR) and the alternating-reduction (AR) methods [15, 16, 19], which are among the fastest *direct* Poisson solvers.

In this paper, we consider in detail only such MGR methods as are related to TR. The characteristic feature of the TR-related MGR methods is the use of a *rotated grid of mesh size* $\sqrt{2}\,h$ between h and $2h$. AR-related MGR methods are of particular interest for anisotropic operators (see Section 6) and for 3D problems.

The original "MGR-0" method was conceived as a fast *approximate* solver [14]; used as an iterative solver, its convergence properties are not satisfactory. The main purpose of this paper is to demonstrate the striking improvement which is achieved by combining MGR-0 with *checkered* (CH) *Gauss-Seidel relaxations*. By this combination one obtains the so called MGR-CH methods, which are analyzed by determining the spectral radii $\rho_h^{\sqrt{2}h}$ and spectral norms $\sigma_h^{\sqrt{2}h}$ for the respective $(h, \sqrt{2}\,h)$-iteration opera-

tors. For example, we prove

$$\rho_h^{\sqrt{2}h} \nearrow \tfrac{1}{2} \quad (h \to 0) \qquad \text{for MGR-0,}$$
$$\rho_h^{\sqrt{2}h} \nearrow \tfrac{2}{27} \quad (h \to 0) \qquad \text{for MGR-CH[1].}$$

In MGR-CH[1] *one* CH-relaxation step (per iteration) is performed; due to the particular algorithmic performance, the computational effort compared to MGR-0 is not essentially enlarged by this. Among all iterative Poisson solvers known so far (see [4], [6], [7], [11], [14], [18], [20]), MGR-CH[1] yields the most efficient algorithm of all (in rate of convergence versus computational effort).

The above results, which refer to the model problem of Poisson's equation, and further, more detailed statements are presented in Section 2 and derived—by simple Fourier-analysis and linear-algebra arguments—in Section 3. In Section 2 we also make some remarks about the relationship of the MGR methods to the algorithms considered by Braess [1]. In particular, the original MGR-0 method (described in a somewhat different way and apparently developed independently) is also analyzed theoretically in [1]. By the use of finite-element arguments and a strengthened Cauchy inequality, Braess obtains the abovementioned bound of $\frac{1}{2}$ (namely, for the energy norm) also for Poisson's equation in more general regions. This is a remarkable theoretical result. The bound of $\frac{1}{2}$—for the $(h, \sqrt{2}\,h)$ version—is, however, of very little practical use; see Sections 4 and 5. Unfortunately, although Braess allows additional CH relaxations in his description of the MGR method, his proof does not take this possibility into account and thus does not explain the essential improvement achieved by it.[1]

In Section 4 we consider the corresponding $(h, 2h)$ versions of various MGR methods. We give spectral radii, spectral norms, and computational-effort quantities, in order to perform efficiency comparisons. This section contains several results which are derived in detail in [20]. In particular, for the main MGR variants the spectral radii for the $(h, 2h)$ versions turn out to be the same as for the $(h, \sqrt{2}\,h)$ versions. Complete multilevel cycles are briefly discussed in Section 5.

This paper has been called a "note" because the MGR idea is—for simplicity—described only for the simplest model problem and because we consider only two MGR algorithms (MGR-0 and MGR-CH[1]) in detail. The special situation (discrete Poisson equation with Dirichlet boundary condi-

[1] In recent papers [2, 3], Braess was able to improve his result by taking into account the effect of additional relaxation steps. For certain convex domains, he obtains not optimal, but good bounds for the respective convergence factors.

tions in the unit square) is exploited in the proofs, as we make use of the knowledge of the eigenfunctions of the discrete Laplace operator. (A more systematical investigation of various standard and nonstandard multigrid methods in connection with different model problems is given in [18, 20].) We want to point out, however, that the MGR principle is more generally applicable, similarly to the MG principle itself. (See Section 6.)

For the description of the methods we prefer—for well-known theoretical and practical reasons [4, 19]—the use of grid functions and difference operators rather than vectors and matrices. Of course, matrix terminology could also have been used in principle.

2. SPECTRAL RADII AND SPECTRAL NORMS FOR THE $(h, \sqrt{2}h)$ VERSIONS OF MGR

In this section we describe the $(h, \sqrt{2}h)$ versions of the MGR-0 and the MGR-CH methods and derive quantitative results for their spectral radii. All descriptions and results refer to the h-discrete model problem of Poisson's equation in the unit square with Dirichlet boundary conditions:

$$\begin{aligned} L_h u_h(P) &= f(P) \qquad (P \in \Omega_h), \\ u_h(P) &= g(P) \qquad (P \in \bar{\Omega}_h \backslash \Omega_h) \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} &\Omega_h = \{P = (x, y) \in \mathbb{R}^2 : x = ih,\ y = jh\,(i, j = 1, 2, \ldots, N-1)\}, \\ &\bar{\Omega}_h = \{P = (x, y) \in \mathbb{R}^2 : x = ih,\ y = jh\,(i, j = 0, 1, \ldots, N)\}; \\ &N = h^{-1} = 2^p, \qquad p \in \mathbb{N}. \end{aligned} \tag{2.2}$$

L_h denotes the usual 5-point approximation of the Laplacian $-\Delta$:

$$L_h = -\Delta_h = \frac{1}{h^2} \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_h .$$

For the description of the algorithms we make the following *formal assumptions*: All approximations of u_h $\left(u_h^{(n)}, u_h^{(n+1)}, \bar{u}_h^{(n)}, \text{etc.}\right)$ are supposed to be

grid functions on $\bar{\Omega}_h$ that satisfy *the exact boundary conditions* $= g(p)$ on $\bar{\Omega}_h \backslash \Omega_h$. Consequently, all "error quantities" occurring $(v_h^{(n)}$ etc.$)$ are also defined on $\bar{\Omega}_h$, with *zero* boundary conditions $(= 0)$ on $\bar{\Omega}_h \backslash \Omega_h$. The right-hand side f and the "defect quantities" $d_h^{(n)}$ etc. naturally are given (and needed) only on Ω_h. For our description it is, however, convenient (at some places) to assume them to be extended to $\bar{\Omega}_h$ by zero, i.e., $= 0$ on $\bar{\Omega}_h \backslash \Omega_h$.

Apart from $\bar{\Omega}_h$, we here introduce the following grids:

$$\bar{\Omega}_h^E = \{P = (x, y) \in \bar{\Omega}_h : x = ih,\ y = jh,\ i + j \text{ even}\}$$

("even points" of $\bar{\Omega}_h$),

$$\bar{\Omega}_h^O = \bar{\Omega}_h \backslash \bar{\Omega}_h^E$$

("odd points" of $\bar{\Omega}_h$).

$\bar{\Omega}_h, \bar{\Omega}_h^E, \bar{\Omega}_h^O$ are shown in Figure 1 $(N = 8)$. Evidently, the checkered grid $\bar{\Omega}_h^E$ can be identified with a (rotated) grid of mesh size $\sqrt{2}\,h$:

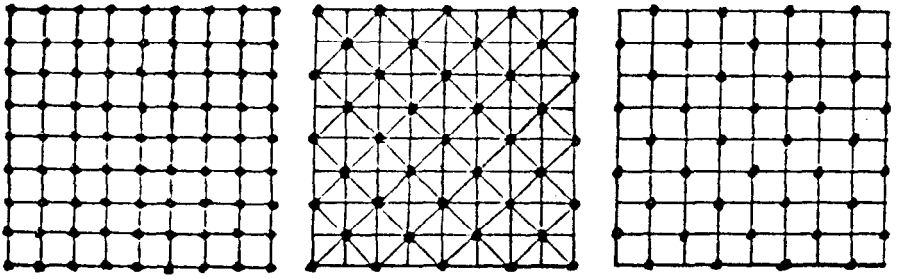
$$\bar{\Omega}_H = \bar{\Omega}_h^E \qquad \text{with} \quad H = \sqrt{2}\,h$$

In the following, we use this notation $\bar{\Omega}_H$ (instead of $\bar{\Omega}_h^E$) as well as

$$\Omega_H := \Omega_h \cap \bar{\Omega}_H \quad \text{and} \quad \Omega_h^O := \Omega_h \cap \bar{\Omega}_h^O.$$

For MGR-0 and MGR-CH[1], we now describe the iteration process

$$u_h^{(n)} \to u_h^{(n+1)} \qquad \text{on } \bar{\Omega}_h.$$



$\bar{\Omega}_h$ $\bar{\Omega}_h^E$ $\bar{\Omega}_h^O$

FIG. 1.

MGR-0 ALGORITHM ((h, H) version)

[I] Compute the defect $d_h^{(n)}$ of $u_h^{(n)}$ on Ω_h:

$$d_h^{(n)}(P) := f(P) - L_h u_h^{(n)}(P) \qquad \text{for} \quad P \in \Omega_h.$$

[II] Restrict $d_h^{(n)}$ to Ω_H, using the weighted restriction operator

$$I_h^H = \frac{1}{8}\begin{bmatrix} & 1 & \\ 1 & 4 & 1 \\ & 1 & \end{bmatrix}_h, \tag{2.3}$$

$$d_H^{(n)}(P) := I_h^H d_h^{(n)}(P) \qquad \text{for} \quad P \in \Omega_H$$

[III] Solve exactly the H-discrete defect equations

$$L_H v_H^{(n)}(P) = d_H^{(n)}(P) \qquad \text{for} \quad P \in \Omega_H$$

with zero Dirichlet boundary conditions. Here L_H is given by

$$L_H = -\Delta_H = \frac{1}{2h^2}\begin{bmatrix} -1 & & -1 \\ & 4 & \\ -1 & & -1 \end{bmatrix}_h. \tag{2.4}$$

[IV] For the $(H \to h)$ transfer of the correction $v_H^{(n)}$, set

$$v_h^{(n)}(P) = v_H^{(n)}(P) \qquad \text{for} \quad P \in \bar{\Omega}_H,$$

and—using these values—solve the h-discrete defect equations

$$L_h v_h^{(n)}(P) = d_h^{(n)}(P) \qquad \text{for} \quad P \in \Omega_h^O \tag{2.5}$$

with zero Dirichlet boundary conditions. [All equations of (2.5) are explicit equations, since for any unknown each of the values $v_h^{(n)}$ of all neighbors of $P \in \Omega_h^O$ is already known: The system can be described by a diagonal matrix.]

[V] Set

$$u_h^{(n+1)} = u_h^{(n)} + v_h^{(n)} \qquad \text{for} \quad P \in \bar{\Omega}_h.$$

We should like to point out that there are several possibilities to construct algorithms that are equivalent to MGR-0. Some of such algorithms require even less operations. (Such an equivalent formulation has been given by

Braess in [1]; see Remark 2.2 for details.) Of course, such modifications should be exploited when the algorithm is implemented (see Section 4). We have preferred the above description because it directly corresponds to the theoretical representation of the iteration operator in Section 3 and because this description is oriented to standard MG-algorithms.

The algorithm MGR-CH[1] described in the following is a combination of MGR-0 with one step of Gauss-Seidel iteration (i.e. successive relaxation with relaxation parameter $\omega = 1$). In the case of checkered ordering of the grid points, which we assume here (also called "red-black" or "chessboardlike" ordering), the Gauss-Seidel operator R_h can be described as a product of two (Jacobi-like) operators R_h^O, R_h^E which belong to Ω_h^O and Ω_h^E, respectively.

MGR-CH[1] ALGORITHM ((h, H) version)

[0] Do one step of Gauss-Seidel iteration with "even-odd checkered" ordering of the grid points:

$$\bar{u}_h^{(n)} = R_h u_h^{(n)} + r_h \qquad \text{on } \bar{\Omega}_h. \tag{2.6}$$

Here R_h is formally defined by

$$R_h = R_h^O \cdot R_h^E, \tag{2.7}$$

with

$$R_h^O w(P) = \begin{cases} \frac{1}{4}\begin{bmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{bmatrix}_h w(P) & \text{for } P \in \Omega_h^O, \\ w(P) & \text{for } P \in \bar{\Omega}_h \backslash \Omega_h^O, \end{cases}$$

$$R_h^E w(P) = \begin{cases} \frac{1}{4}\begin{bmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{bmatrix}_h w(P) & \text{for } P \in \Omega_H, \\ w(P) & \text{for } P \in \bar{\Omega}_h \backslash \Omega_H, \end{cases}$$

and r_h is given by

$$r_h(P) = \begin{cases} \frac{h^2}{4} f(P) & \text{for } P \in \Omega_H, \\ \frac{1}{4}\begin{bmatrix} & 1 & \\ 1 & 4 & 1 \\ & 1 & \end{bmatrix}_h \frac{h^2}{4} f(P) & \text{for } P \in \Omega_h^O, \\ 0 & \text{for } P \in \bar{\Omega}_h \backslash \Omega_h. \end{cases}$$

[I] through [V]: As in MGR-0 with $\bar{u}_h^{(n)}$ instead of $u_h^{(n)}$.

REMARK 2.1. In this description, we have—for simplicity—explained the MGR-CH[1] method by reducing it to MGR-0. If MGR-CH[1] is regarded by itself, there are two essential simplifications compared to MGR-0 which should of course be exploited in the algorithmic performance of MGR-CH[1]. These simplifications are due to the fact that the even-odd relaxation $R_h^O R_h^E$ in [0] implies

$$d_h^{(n)} = 0 \qquad \text{on } \Omega_h^0. \tag{2.8}$$

As a first consequence, the application of

$$I_h^H = \frac{1}{8}\begin{bmatrix} & 1 & \\ 1 & 4 & 1 \\ & 1 & \end{bmatrix}_h$$

in (2.3) degenerates to a multiplication by $\frac{1}{2}$. As a second consequence, the right-hand side of (2.5) is zero. Therefore the use of the discrete equation in (2.5) degenerates to bilinear interpolation (also see Remark 2.3).

Evidently, the two processes MGR-0 and MGR-CH[1] can be characterized by

$$u_h^{(n+1)} = M_h^H u_h^{(n)} + s_h \qquad (n = 0,1,2,\dots) \tag{2.9}$$

with linear iteration operators M_h^H. For the explicit representation of M_h^H see Section 3. Clearly,

$$M_h^H[\text{MGR-CH}[1]] = M_h^H[\text{MGR-0}] \cdot R_h.$$

The asymptotic convergence properties of the two methods are characterized by

THEOREM 1. *For the spectral radii $\rho_h^H = \rho(M_h^H)$ of the iteration operators we obtain the following upper limits independent of h:*

$$\rho_h^H \nearrow \tfrac{1}{2} \quad (h \to 0) \qquad \textit{for MGR-0}, \tag{2.10}$$

$$\rho_h^H \nearrow \tfrac{2}{27} \quad (h \to 0) \qquad \textit{for MGR-CH[1]}. \tag{2.11}$$

The following supplements refer to the algorithms MGR-CH[ν]. These are obtained from MGR-CH[1] by performing $\nu \geqslant 1$ steps of checkered Gauss-

Seidel iterations in [0] instead of only one step, whereas all other components of MGR-CH[1] remain unchanged. The iteration operator then is given by

$$M_h^H[\text{MGR-CH}[\nu]] = M_h^H[\text{MGR-0}] \cdot R_h^\nu. \tag{2.12}$$

Also for this more general case the upper limits of $\rho_h^H = \rho_h^H[\nu]$ can be determined exactly. If ν becomes larger, the spectral radii, of course, decrease, whereas the computational effort per iteration step increases. In fact, the efficiency of the MGR-CH[ν] methods (convergence rate versus computational effort) turns out to be best for $\nu = 1$. (These efficiency aspects are discussed in detail in Section 4.)

SUPPLEMENT 1. *If* $\nu \geqslant 1$, *we have*

$$\rho_h^H[\nu] \nearrow \rho^*[\nu] = \frac{1}{2} \frac{(2\nu)^{2\nu}}{(2\nu+1)^{2\nu+1}} \quad (h \to 0) \qquad \textit{for MGR-CH}[\nu]. \tag{2.13}$$

For $\nu \to \infty$, $\rho^*[\nu]$ *behaves like*

$$\nu\rho^*[\nu] \to \frac{1}{4e}.$$

Whereas the spectral radius $\rho(M)$ characterizes the *asymptotic* convergence properties of the respective iteration process, the spectral norm

$$\sigma(M) = \|M\|_{\text{Sp}} = \sqrt{\rho(MM^*)}$$

gives the $\|\ \|_2$-error reduction factor per iteration step. One obtains

SUPPLEMENT 2. *For* $h \to 0$, *the spectral norms* $\sigma_h^H[\nu]$ *of the* MGR-0 *and* MGR-CH[ν] *methods* ($\nu \geqslant 1$) *converge to the following upper limits*:

$$\sigma_h^H \nearrow \tfrac{1}{2} \qquad \text{for MGR} - 0, \tag{2.14}$$

$$\sigma_h^H[\nu] \nearrow \sigma^*[\nu] = \frac{1}{8 \times 2^{2\nu-1}} (1-\eta^*)(1+\eta^*)^{2\nu-1} \sqrt{4+(1+\eta^*)^2}$$

$$\textit{for MGR-CH}[\nu], \tag{2.15}$$

TABLE 1
SPECTRAL RADII AND SPECTRAL NORMS FOR MGR-CH[ν]

	$\rho^*[\nu]$	$\sigma^*[\nu]$
MGR-0	0.5	0.5
MGR-CH[1]	0.074	0.141
MGR-CH[2]	0.041	0.066
MGR-CH[3]	0.028	0.044

where $\eta^* = \eta^*[\nu]$ *is the only real zero of the cubic equation*

$$(2\nu+1)\eta^3 + (2\nu+3)\eta^2 + (6\nu+3)\eta + (9-10\nu) = 0.$$

Table 1 contains the limits $\rho^*[\nu]$ and $\sigma^*[\nu]$ (three decimals) for some values of ν.

Theorem 1 and the two supplements are proved in the following section. The proofs, which are based on Fourier-analysis arguments, are elementary but technically somewhat complicated.

We here make some remarks about related approaches and special features of MGR.

REMARK 2.2. As pointed out already in the introduction, there is a very close relationship between the MGR methods considered above and the methods described by Braess in [1]. In detail, we find: The (h, H) version of MGR-0 is equivalent to Braess's Algorithm 2.1 in [1] with $\nu = 0$. In the cases $\nu = 1$ and $\nu = 2$, Braess's algorithms are very similar to MGR-CH[ν]; indeed, the only difference is that the Gauss-Seidel relaxation operator R_h in [1] is defined by $R_h = R_h^E \cdot R_h^O$ instead of (2.7) (odd-even checkered relaxation instead of even-odd). This slight difference has no influence on the spectral radii $\rho_h^H[\nu]$ (in the case of the model problem). Therefore, by arguments similar to those above, one also obtains for Braess's methods

$$\rho^*[0] = \tfrac{1}{2}, \quad \rho^*[1] = \tfrac{2}{27} \approx 0.074, \quad \rho^*[2] = \tfrac{128}{3125} \approx 0.041 \qquad (\text{for} \quad h \to 0).$$

Braess shows in [1] for his methods only[2]

$$\rho_h^H[\nu] \leqslant \tfrac{1}{2} \qquad (\text{for} \quad \nu = 0,1,2 \text{ and arbitrary } h). \tag{2.16}$$

[2]Cf. Footnote 1.

REMARK 2.3. For the general concept of MGR, we regard the use of the discrete equations in the (H, h) transfer [cf. (2.5)] as essential, although in MGR-CH it degenerates to bilinear interpolation from Ω_H to Ω_h, i.e. to the application of

$$\frac{1}{4}\begin{bmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{bmatrix}_h .$$

Of course, if the relaxation operator R_h is chosen in a different way (for example as in Braess's methods), the use of the discrete equations cannot simply be replaced by bilinear interpolation.

There are several interesting interpretations of the use of the discrete equations. For example, it can be regarded as (and be replaced by; see [1]) "half a step" of relaxation, which means that there is some "inherent smoothing" in this process. This explains why for MGR-0 one obtains the spectral radius (upper limit) $\frac{1}{2}$ instead of 1. (The value 1 is characteristic for the usual coarse-grid correction operators of standard MG methods [18]. One would obtain it also for MGR-0 if the use of the discrete equation were replaced by bilinear interpolation there.) For a formal description of the use of the discrete equations as an appropriate relaxation process see [20].

Furthermore, the error in the (H, h) transfer is of order $O(h^2)$ in the case of bilinear interpolation, but of order $O(h^4)$ if the discrete equations are used.

3. PROOF OF THEOREM 1 AND THE TWO SUPPLEMENTS

The basis of the following proof is the fact that the eigenfunctions of $-\Delta_h$ form simple (i.e. two-dimensional) subspaces which are invariant under the iteration operators M_h^H of MGR-0 and MGR-CH[ν]. The proof is divided in several parts.

(1) Representation of M_h^H [MGR-0]

For the representation of M_h^H, we first note that the (H, h) transfer in MGR-0 (part [IV] in the algorithm; see Section 2) can equivalently be written as

$$v_h^{(n)}(P) = \begin{cases} I_H^h v_H^{(n)}(P) + \dfrac{h^2}{4} \mathring{I}_H d_h^{(n)}(P) & \text{for } P \in \Omega_h \\ 0 & \text{for } P \in \overline{\Omega}_h \backslash \Omega_h \end{cases}$$

where I_H^h denotes the bilinear interpolation operator

$$I_H^h w(P) := \begin{cases} w(P) & \text{for } P \in \Omega_H, \\ \frac{1}{4}\begin{bmatrix} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{bmatrix}_h w(P) & \text{for } P \in \Omega_h^O, \end{cases}$$

and

$$\mathring{I}_H w(P) := \begin{cases} 0 & \text{for } P \in \Omega_H, \\ w(P) & \text{for } P \in \Omega_h^O \end{cases}$$

(zero on Ω_H, identity on Ω_h^O).

Using this representation, one can easily recognize from the description of MGR-0 that M_h^H is given by

$$M_h^H[\text{MGR-0}] = I_h - I_H^h L_H^{-1} I_h^H L_h - \frac{h^2}{4} \mathring{I}_H L_h. \tag{3.1}$$

(Compared to standard MG methods [4, 9, 18], the MGR-0 iteration operator contains the additional term $h^2/4\mathring{I}_H L_h$ which reflects the use of the discrete equations in the (H, h) transfer.)

(2) Notation

By

$$\mathbb{G}(\Omega_h), \mathbb{G}(\Omega_H), \mathbb{G}(\Omega_h^O)$$

we denote the respective spaces of grid functions (on $\Omega_h, \Omega_H, \Omega_h^O$), which are additionally assumed to satisfy zero boundary conditions. By ψ_{kl} $(k, l = 1, 2, \ldots, N-1)$ we denote the discrete eigenfunctions of the model operator $-\Delta_h$ (with Dirichlet boundary conditions):

$$\psi_{kl}(x, y) = \sin(k\pi x)\sin(l\pi y), \qquad (x, y) \in \Omega_h.$$

These ψ_{kl} form a basis of $\mathbb{G}(\Omega_h)$.

Given $k, l \in \mathbb{N}_{N-1} = \{1, 2, \ldots, N-1\}$, let $\bar{k} = N - k$, $\bar{l} = N - l$. From the relations

$$\tfrac{1}{2}(\psi_{kl} + \psi_{\bar{k}\bar{l}})(x, y) = \begin{cases} \psi_{kl}(x, y) & \text{if } (x, y) \in \Omega_H, \\ 0 & \text{if } (x, y) \in \Omega_h^O, \end{cases}$$

$$\tfrac{1}{2}(\psi_{kl} - \psi_{\bar{k}\bar{l}})(x, y) = \begin{cases} 0 & \text{if } (x, y) \in \Omega_H, \\ \psi_{kl}(x, y) & \text{if } (x, y) \in \Omega_h^O, \end{cases}$$

it then follows that

$$\psi'_{kl} := \tfrac{1}{2}(\psi_{kl} + \psi_{\bar{k}\bar{l}})|_{\Omega_H} \qquad \text{for} \quad (k,l) \in J'$$

is a basis of $\mathfrak{G}(\Omega_H)$ and that

$$\bar{\psi}'_{kl} := \tfrac{1}{2}(\psi_{kl} - \psi_{\bar{k}\bar{l}})|_{\Omega_H^O} \qquad \text{for} \quad (k,l) \in \mathbb{N}_{N-1}^2 \setminus J'$$

is a basis of $\mathfrak{G}(\Omega_h^O)$. Here the index set J' is given by

$$J' = \left\{ (k,l) \in \mathbb{N}_{N-1}^2 : k + l \leqslant \left\{ \begin{array}{ll} N & \text{if} \quad 2k \leqslant N \\ N-1 & \text{if} \quad 2k > N \end{array} \right\} \right\}. \tag{3.2}$$

By E_{kl} we denote the spaces

$$E_{kl} = \operatorname{span}\langle \psi_{kl}, \psi_{\bar{k}\bar{l}} \rangle, \qquad (k,l) \in J'. \tag{3.3}$$

The E_{kl} are two-dimensional except for $k = l = N/2$, when E_{kl} is one-dimensional.

(3) Proof of (2.10): ρ_h^H for MGR-0

We first show that the E_{kl} are invariant under M_h^H. Setting $c_k := \cos k\pi h$, $c_{kl} := \frac{1}{2}(c_k + c_l)$, one has [for $(k,l) \in J'$]

$$L_h : \begin{cases} \psi_{kl} \to \dfrac{4}{h^2}(1 - c_{kl})\psi_{kl}, \\ \psi_{\bar{k}\bar{l}} \to \dfrac{4}{h^2}(1 + c_{kl})\psi_{\bar{k}\bar{l}}, \end{cases}$$

$$I_h^H : \begin{cases} \psi_{kl} \to \frac{1}{2}(1 + c_{kl})\psi'_{kl}, \\ \psi_{\bar{k}\bar{l}} \to \frac{1}{2}(1 - c_{kl})\psi'_{kl}, \end{cases}$$

$$L_H^{-1} : \psi'_{kl} \to \frac{h^2}{2(1 - c_k c_l)}\psi'_{kl},$$

$$I_H^h : \psi'_{kl} \to \tfrac{1}{2}(1 + c_{kl})\psi_{kl} + \tfrac{1}{2}(1 - c_{kl})\psi_{\bar{k}\bar{l}},$$

$$\mathring{I}_H : \begin{cases} \psi_{kl} \to -\bar{\psi}'_{\bar{k}\bar{l}} \\ \psi_{\bar{k}\bar{l}} \to \bar{\psi}'_{\bar{k}\bar{l}} \end{cases} \qquad \text{if} \quad (k,l) \neq \left(\frac{N}{2}, \frac{N}{2}\right),$$

$$\psi_{N/2,\, N/2} \to 0.$$

These relations imply the invariance of E_{kl} under M_h^H. Furthermore, with respect to the orthogonal basis

$$\{\psi_{kl}, \psi_{\bar{kl}} : (k,l) \in J'\}, \tag{3.4}$$

we obtain that M_h^H is orthogonally equivalent to $(M_{kl} : (k,l) \in J')$. Here M_{kl} denote the matrices

$$M_{kl} = \frac{(c_k - c_l)^2}{8(1 - c_k c_l)} \begin{bmatrix} 1 + c_{kl} & 1 + c_{kl} \\ 1 - c_{kl} & 1 - c_{kl} \end{bmatrix}, \qquad (k,l) \neq \left(\frac{N}{2}, \frac{N}{2}\right), \tag{3.5}$$

and the 1×1 matrix

$$M_{N/2,N/2} = 0.$$

Clearly,

$$\rho(M_h^H) = \max_{(k,l) \in J'} \rho(M_{kl}) \quad \text{and} \quad \rho(M_{kl}) = \frac{(c_k - c_l)^2}{4(1 - c_k c_l)}.$$

If (k,l) ranges over J', then (c_k, c_l) varies in the triangle

$$D = \{(\xi, \eta) : -1 \leqslant \xi \leqslant 1, \ -\xi \leqslant \eta \leqslant 1\} \tag{3.6}$$

(see Figure 2). For $h \to 0$, we finally have

$$\rho(M_h^H) \nearrow \max_{(\xi,\eta) \in D} \frac{(\xi - \eta)^2}{4(1 - \xi\eta)} = \frac{1}{2}.$$

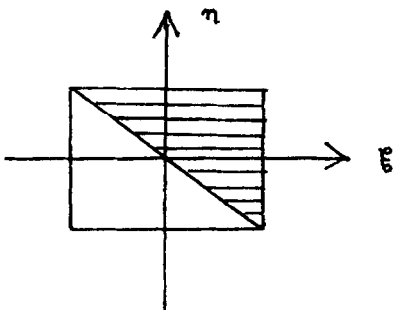


FIG. 2.

The maximum is attained for $(\xi, \eta) = (1, -1)$.

(4) Proof of (2.11): ρ_h^H for MGR-CH[ν]

The M_h^H operator of MGR-CH[ν] is given by

$$M_h^H = \left[I_h - I_H^h L_H^{-1} I_h^H L_h - \frac{h^2}{4} \mathring{I}_H L_h \right] R_h^\nu \tag{3.7}$$

[cf. (2.12),(3.1)]. The spaces E_{kl} $[(k,l) \in J']$ are also invariant under this operator M_h^H. Namely, under $R_h = R_h^O R_h^E$, the ψ_{kl} and $\psi_{\bar{k}\bar{l}}$ are transformed in the following way:

$$R_h: = \begin{cases} \psi_{kl} \to c_{kl}\left[\tfrac{1}{2}(1+c_{kl})\psi_{kl} + \tfrac{1}{2}(1-c_{kl})\psi_{\bar{k}\bar{l}}\right], \\ \psi_{\bar{k}\bar{l}} \to -c_{kl}\left[\tfrac{1}{2}(1+c_{kl})\psi_{kl} + \tfrac{1}{2}(1-c_{kl})\psi_{\bar{k}\bar{l}}\right]. \end{cases}$$

Finally, instead of (3.5), we now obtain the following representation of M_h^H[MGR-CH[ν]] with respect to (3.4):

$$M_{kl}[\nu] = c_{kl}^{2\nu-1} \frac{(c_k - c_l)^2}{8(1-c_k c_l)} \begin{bmatrix} 1+c_{kl} & -(1+c_{kl}) \\ 1-c_{kl} & -(1-c_{kl}) \end{bmatrix}, \qquad (k,l) \neq \left(\frac{N}{2}, \frac{N}{2}\right), \tag{3.8}$$

$$M_{N/2,\,N/2}[\nu] = 0.$$

Consequently,

$$\rho_h^H[\nu] = \max_{(k,l) \in J'} \rho(M_{kl}[\nu]) \nearrow \max_{(\xi,\eta) \in D} g_\nu(\xi, \eta) \qquad (h \to 0)$$

with

$$g_\nu(\xi, \eta) = \frac{(\xi - \eta)^2}{4(1-\xi\eta)} \left(\frac{\xi + \eta}{2}\right)^{2\nu}$$

and D as in (3.6).

By monotonicity arguments (for fixed ν) it can easily be shown that g_ν attains its maximum on the line $\xi = 1$. Hence it can be found by differentiation. We obtain

$$\max_{(\xi,\eta) \in D} g_\nu(\xi, \eta) = g_\nu\left(1, \frac{2\nu-1}{2\nu+1}\right) = \frac{1}{2} \frac{(2\nu)^{2\nu}}{(2\nu+1)^{2\nu+1}}.$$

In particular, for $\nu = 1$ we obtain the value of $\frac{2}{27}$ at $(\xi, \eta) = (1, \frac{1}{3})$.

(5) Proof of Supplement 2: σ_h^H for MGR-0, MGR-CH[ν]

Here one has to determine

$$\sigma_h^H = \max_{(k,l)\in J'} \sigma(M_{kl}), \qquad \sigma(M_{kl}) = \sqrt{\rho(M_{kl}M_{kl}^*)}$$

for the matrices M_{kl} in (3.5) (MGR-0) and in (3.8) (MGR-CH[ν]), respectively. In the case of MGR-0, one obtains for $h \to 0$

$$\sigma_h^H \nearrow \max_{(\xi,\eta)\in D} \frac{(\xi-\eta)^2}{8(1-\xi\eta)}\sqrt{4+(\xi+\eta)^2} = \frac{1}{2} \qquad [\text{at} \quad (\xi,\eta) = (1,-1)]$$

with D as in (3.6).

In the case of MGR-CH[ν] ($\nu \geqslant 1$), one has

$$\sigma_h^H[\nu] \nearrow \max_{(\xi,\eta)\in D} \frac{(\xi-\eta)^2}{8(1-\xi\eta)}\left(\frac{\xi+\eta}{2}\right)^{2\nu-1}\sqrt{4+(\xi+\eta)^2}$$

By simple monotonicity arguments one recognizes that the maximum is attained at $\xi = 1$. Differentiation with respect to η finally yields the formula (2.15).

4. COMPARISON OF SEVERAL $(h,2h)$ METHODS: CONVERGENCE PROPERTIES AND COMPUTATIONAL EFFORT

The analysis of the (h, H) versions of MGR-0 and MGR-CH[ν] considered in the preceeding sections gives no information about the efficiency of these algorithms compared to other ones as long as the computational effort is not taken into account. For the following comparison, we consider the $(h,2h)$ versions of the respective methods instead of the (h, H) versions, as most multigrid methods are recursively defined by means of $(h,2h)$.

Table 2 gives a survey of

$$\rho_h^{2h} = \rho(M_h^{2h}), \qquad \sigma_h^{2h} = \sigma(M_h^{2h}), \qquad \theta_h^{2h}.$$

Here M_h^{2h} denotes the $(h,2h)$ iteration operator. By θ_h^{2h}, we give the number of arithmetic operations ($\times$ and $+$) per grid point of Ω_h and per iteration

step, *excluding* the solution of the Ω_{2h}-equations. (If the Ω_{2h}-equations are solved by a recursively defined MGR cycle of V- or of W-type, θ_h^{2h} has to be multiplied by a factor of $\frac{4}{3}$ or 2, respectively, to obtain the total amount of work per complete multigrid iteration step. See Section 5 for some details.)

The MGR methods MGR-CH[ν, ν'] considered in this section are—roughly speaking—characterized by the following components for each iteration step $u_h^{(n)} \to u_h^{(n+1)}$:

[0] ν steps of Gauss-Seidel even-odd checkered relaxations on Ω_h,
[I] defect restriction to Ω_H,
[0′] ν' steps of Gauss-Seidel even-odd checkered relaxation on Ω_H (for the correction, with starting vector 0),
[II] defect restriction to Ω_{2h},
[III] exact solution on Ω_{2h},
[IV] $(2h \to H)$ transfer using the H-discrete equations,
[V] $(H \to h)$ transfer using the h-discrete equations.

Instead of a detailed description of these algorithms we give here only a representation of the corresponding iteration operators $M_h^{2h} = M_h^{2h}$[MGR-CH[ν, ν']]:

$$M_h^{2h} = \left[I_h - I_H^h \left(I_H - M_H^{2h} \right) L_H^{-1} I_h^H L_h - \frac{h^2}{4} \mathring{I}_H L_h \right] R_h^\nu$$

with

$$M_H^{2h} = \left[I_H - I_{2h}^H L_{2h}^{-1} I_H^{2h} L_H - \frac{h^2}{2} \mathring{I}_{2h} L_H \right] R_H^\nu;$$

here the operators I_{2h}^H, $L_{2h} = -\Delta_{2h}$, I_H^{2h}, $\mathring{I}_{2h}$, R_H are defined with respect to Ω_{2h}, Ω_H in exactly the same manner as the corresponding operators I_H^h, $L_H = -\Delta_H$, I_h^H, $\mathring{I}_H$, R_h with respect to Ω_H, Ω_h (see Section 2). (In the case $\nu = \nu'$ the $(h, 2h)$ version of MGR-CH[ν, ν'] could have been defined also recursively based on the (h, H) version of MGR-CH[ν].)

Interpreting M_h^{2h} as an approximation of M_h^H in (3.7), one can represent it also in the following way:

$$M_h^{2h} = M_h^H + I_H^h M_H^{2h} L_H^{-1} I_h^H L_h R_h^\nu.$$

REMARK 4.1. In the cases $\nu \geqslant 1$ and $\nu' \geqslant 1$ there are again inherent simplifications in the above algorithms due to the fact that the corresponding defects are zero on Ω_h^0 and Ω_H^0, respectively $\left(\mathring{I}_H L_h R_h = 0, \mathring{I}_{2h} L_H R_H = 0 \right)$. The corresponding discrete equations in [IV] and [V], respectively, are not really used, but replaced by bilinear interpolation, in these cases. Notice, however, that in the case $\nu' = 0$ the use of the discrete equation in the $2h \to H$ transfer is essential.

REMARK 4.2. For $\nu = \nu'$, one obtains the $(h, 2h)$ versions of MGR-CH[ν]. The methods MGR-CH[1,0] and MGR-CH[1,1] have been mentioned already in [7]; they were called respectively MGR-CH 1 and MGR-CH 2 there.

The spectral radii ρ_h^{2h} and spectral norms σ_h^{2h} for the MGR-CH[ν, ν'] methods have been determined by theoretical considerations similar to those of the last section. The only difference is that for the $(h, 2h)$ versions considered here the spaces

$$\operatorname{span}\langle \psi_{kl}, \psi_{k\bar{l}}, \psi_{\bar{k}l}, \psi_{\bar{k}\bar{l}} \rangle \qquad \left(k, l = 1, \ldots, \frac{N}{2} - 1\right)$$

remain invariant under M_h^{2h}, but the (two-dimensional) spaces E_{kl} in (3.3) do not. Therefore, the corresponding Fourier analysis, in general, leads to 4×4 matrices instead of 2×2 matrices, and is technically somewhat more complicated (see [18], [20] for details). The values for ρ_h^{2h} and σ_h^{2h} in Table 2 refer to the case $N = h^{-1} = 64$; they are, however, nearly independent of h in the sense that the first two decimals of these values do not change if h becomes smaller.

Only for comparison, we also consider $(h, 2h)$ versions of a standard multigrid technique, namely LEX(ν,0)-INJ, roughly explained in the following. (Such standard methods have formerly been recommended for the Poisson equation by A. Brandt [4]; also see the corresponding remarks in [7], [18].) Here the smoothing is performed by ν steps of Gauss-Seidel point relaxation with lexicographic (row- or columnwise) ordering of the Ω_h-points. Afterwards, the $(h \to 2h)$ restriction is carried out simply by $I_h^{2h} v(P) := v(P)$ $\left(P \in \bar{\Omega}_{2h}\right)$ ("injection"). On the $2h$-grid, $L_h = -\Delta_h$ is approximated by $L_{2h} = -\Delta_{2h}$. For the $(2h \to h)$ transfer, bilinear interpolation is used. The values for ρ_h^{2h} corresponding to LEX(ν,0)-INJ given in Table 2 are only approximate ones (marked by *); they have been calculated by the so-called "local mode analysis" (Brandt [4]). By this one can also recognize that the spectral norms σ_h^{2h} of these methods behave like $O(h^{-2})$ (for $h \to 0$).

In order to judge the efficiency of the various methods in Table 2 it is, of course, necessary to look at both the spectral radii ρ_h^{2h} (and norms σ_h^{2h}) and the computational effort θ_h^{2h}. We have noted our evaluation of the methods in the last column of Table 2. (As a measure of efficiency we have used here

$$\text{eff} := \left[\rho_h^{2h}\right]^{\alpha} \qquad \text{with} \quad \alpha = 10/\theta_h^{2h},$$

i.e. an asymptotic convergence factor per $10h^{-2}$ arithmetic operations.)

In addition to the numbers given in Table 2 we remark that one obtains (numerically)

$$\left.\begin{aligned}\rho_h^{2h}[\text{MGR-CH}[\nu,\nu']] &= \rho_h^{2h}[\text{MGR-CH}[\nu,\nu]]\\ \sigma_h^{2h}[\text{MGR-CH}[\nu,\nu']] &= \sigma_h^{2h}[\text{MGR-CH}[\nu,\nu]]\end{aligned}\right\} \quad \text{if} \quad \nu' \geqslant \nu,$$

so that it is of no use to do more CH relaxations on Ω_H than on Ω_h.

With respect to the considerations in the next section concerning "complete cycles" the following observation is very important. It refers to the respective (h, H) and $(h, 2h)$ versions of MGR-CH[ν], i.e. $\nu = \nu'$.

REMARK 4.3. The numerical calculation of ρ_h^H (and σ_h^H) for MGR-CH[ν] and ρ_h^{2h} (and σ_h^{2h}) for MGR-CH[ν,ν] yields the results given in Table 3 $\left(h = \frac{1}{64}\right)$. We see that *the spectral radii and the spectral norms are not*

TABLE 2[a]

Method	ρ_h^{2h}	σ_h^{2h}	θ_h^{2h} ×	θ_h^{2h} +	Eff.	Comment
MGR-CH[0,0]	0.499	0.499	2	7	0.46	Not competitive
MGR-CH[1,1]	0.074	0.141	3	9.75	*0.13*	Very efficient
MGR-CH[2,2]	0.041	0.066	4.5	14.25	0.18	
MGR-CH[3,3]	0.028	0.043	6	18.75	0.24	
MGR-CH[1,0]	0.096	0.169	2.5	8.5	*0.12*	Very efficient
MGR-CH[2,0]	0.062	0.100	3.5	11.5	0.16	
MGR-CH[3,0]	0.047	0.071	4.5	14.5	0.20	
MGR-CH[2,1]	0.047	0.071	4	12.75	0.16	
MGR-CH[3,1]	0.037	0.055	5	15.75	0.20	
MGR-CH[3,2]	0.028	0.043	5.5	17.25	0.21	
LEX(1,0)-INJ	0.447*	(∞)	2.5	6.5	0.41	Less efficient
LEX(2,0)-INJ	0.2 *	(∞)	3.5	10.5	0.32	
LEX(3,0)-INJ	0.089*	(∞)	4.5	14.5	0.28	
LEX(4,0)-INJ	0.042*	(∞)	5.5	18.5	0.27	
LEX(5,0)-INJ	0.028*	(∞)	6.5	22.5	0.29	

[a]As already mentioned in Section 2, there are several possibilities for implementing the MGR-CH[ν,ν'] method, resulting in different operation counts. Some of such algorithmic simplifications (cf. Remark 2.1 etc.) have been exploited in giving the above numbers for θ_h^{2h}.

enlarged by the step from H to 2h. For the significance of this phenomenon for practical purposes, see Section 5.

However, if the $(h,4h)$ versions of MGR-CH$[\nu]$ are considered ("V-cycles"; see next section), one obtains larger values $\rho_h^{4h}[\nu]$ than the above $\rho_h^{2h}[\nu]$; for instance

$$\rho_h^{4h}[0] = 0.674, \quad \rho_h^{4h}[1] = 0.093 \qquad \left(\text{for} \quad h = \tfrac{1}{64}\right).$$

See [20] for details.

5. COMPLETE CYCLES

The analysis of the previous sections (and the comparison of computational effort) refers to the $(h,2h)$ and the (h, H) versions of the respective methods. Such two-level versions are, of course, not used in practice; they serve only as a basis for "complete" (multilevel) iteration cycles or for "full" multigrid algorithms [18]. On the basis of $(h,2h)$ versions, such algorithms can be defined (recursively or adaptively) in the MGR context in exactly the same manner as the standard MG techniques. We shall not explain these straightforward procedures here. We should like, however, to point out the significant difference between complete cycles based on $(h,2h)$ and on (h, H).

The simplest $(h,2h)$ *recursion* of a complete cycle is characterized by the number γ of MGR (or MG) iterations which are performed on the next coarser grid (mesh size $2h^*$) to solve approximately the h^*-grid equations $(h^* = h, 2h, 4h, \ldots, h_0/2)$. (In the case of the model problem the coarsest mesh size can be chosen as $h_0 = \frac{1}{2}$.)

Let $\sigma_h(\gamma)$ denote the spectral norm of the iteration operator which describes such a complete cycle. By a simple recursive estimate, one can obtain an h-independent bound for $\sigma_h(\gamma)$, provided $\gamma \geqslant 2$ and σ_h^{2h} (cf. the previous section) is sufficiently small $\left(\text{for example } \sigma_h^{2h} \leqslant \frac{1}{4}\right)$ [18].

TABLE 3

ν	ρ_h^H	ρ_h^{2h}	σ_h^H	σ_h^{2h}
0	0.5	0.5	0.5	0.5
1	0.074	0.074	0.141	0.141
2	0.047	0.047	0.066	0.066
3	0.028	0.028	0.043	0.043

TABLE 4

γ	1	2	3	4
$\theta_h(\gamma) \leqslant$	$\frac{4}{3}\theta_h^{2h}$	$2\theta_h^{2h}$	$4\theta_h^{2h}$	$\theta_h^{2h} O(\log h^{-1})$

In the case $\gamma = 2$ we speak of a complete cycle of "W-type." In the case $\gamma = 1$, which we refer to as a complete cycle of "V-type," the corresponding estimation technique yields no h-independent bound for $\sigma_h(1)$.[3] Therefore, from a theoretical point of view, a complete cycle with $\gamma \geqslant 2$ is more satisfactory, at least simpler to handle.

On the other hand, the computational effort for a complete cycle, denoted by $\theta_h(\gamma)$, is proportional to θ_h^{2h} and thus proportional to h^{-2} (points of the finest grid) only if $\gamma \leqslant 3$. Namely, one has the estimates shown in Table 4. (These estimates are easily obtained by a geometric-series relation Σq^k, taking into account the number of grid points of $\Omega_{2^k h}$—yielding a convergence factor of $q = \gamma/4$.) So in the case of an $(h, 2h)$ recursion, the demands $\gamma \geqslant 2$ and $\gamma \leqslant 3$ seem to be natural from the theoretical and from the practical point of view, respectively. With respect to efficiency, $\gamma = 2$ is preferable.

As pointed out also by Braess [1], the situation is essentially different for complete cycles (of MGR type), which are defined in the simplest way by a (h, H) recursion. Let γ' now denote the number of MGR iterations which are performed on the next coarser grid (*mesh size* $\sqrt{2}\, h^*$) to solve approximately the h^*-grid equations ($h^* = \sqrt{2}^k h$, $k = 0, 1, 2, \ldots$). If one chooses the same γ' for all grids Ω_{h^*}, practical considerations similar to those above yield $\gamma' \leqslant 1$ (instead of $\gamma \leqslant 3$) in order to restrict the total computational work to be proportional to h^{-2}. (In the corresponding geometric series relation Σq^k, one here has to do with $q = \gamma'/2$ instead of $\gamma/4$.)

On the other hand, the theoretical complications for the V-cycle described above are, of course, present also in the framework of the (h, H) recursion. This is the reason why in our opinion it is important to obtain small h-independent bounds for ρ_h^{2h} and σ_h^{2h} for the corresponding $(h, 2h)$ MGR methods (cf. Remark 4.3). Nevertheless, the values for ρ_h^H, σ_h^H (as derived in Section 2) give some theoretical insight into the principal convergence behavior of the different approaches.

[3] By use of a modified argument, recently Braess [3] has obtained h-independent bounds for the convergence factors of certain V-cycle MG methods. In the meantime, this idea has been adopted by several other authors. In all these theoretical approaches, restrictive assumptions have to be made.

6. ANISOTROPIC OPERATORS AND OTHER APPLICATIONS

As already pointed out in the introduction, the MGR principle is not at all restricted to such special situations as the model problem considered in the previous sections. As an example for a more general practical application of MGR, we only mention the program MG 01 [17, 18, 7] for

$$\begin{aligned} -\Delta u(P)+c(P)u(P)&=f(P), \qquad P\in\Omega,\\ u(P)&=g(P), \qquad P\in\partial\Omega, \end{aligned}$$

on general 2D domains Ω. In this program, which is based on a simple multigrid cycle and the full-MG technique [4], all MG components can easily be replaced by corresponding MGR components (e.g. CH(1,1)-FW [7] can be replaced by MGR-CH[1,1]).

Furthermore, with respect to more general problems, the MGR concept has several additional advantages over standard MG techniques. A very simple example for this is the above *Helmholtz equation* (with constant or variable $c \geqslant 0$). The $(h \to H)$-transfer operator naturally suggested by the total-reduction principle is

$$I_h^H = \frac{1}{8+ch^2}\begin{bmatrix} & 1 & \\ 1 & 4+ch^2 & 1 \\ & 1 & \end{bmatrix}_h$$

instead of (2.3). Indeed, the use of this operator yields smaller spectral radii for the corresponding MGR methods than (2.3).

We want to discuss the important case of an *anisotropic operator*

$$Lu = -\varepsilon u_{xx} - u_{yy} \qquad \text{with} \quad 0<\varepsilon\neq 1 \tag{6.1}$$

in some more detail. For simplicity, we consider the following discrete problem:

$$\begin{aligned} L_h u_h(P)&=f(P) \qquad (P\in\Omega_h),\\ u_h(P)&=g(P) \qquad (P\in\bar{\Omega}_h\backslash\Omega_h), \end{aligned} \tag{6.2}$$

where Ω_h is given by (2.2) [square grid of mesh size h in $\Omega=(0,1)^2$] and L_h is

the ordinary 5-point approximation of L in (6.1):

$$L_h = \frac{1}{h^2}\begin{bmatrix} & -1 & \\ -\varepsilon & 2+2\varepsilon & -\varepsilon \\ & -1 & \end{bmatrix}_h .$$

For the problem (6.2), it is well known [17] that pointwise (CH) relaxation has unsatisfactory smoothing properties if $\varepsilon \ll 1$ (or $\varepsilon \gg 1$). These difficulties are easily overcome by line relaxation, or—still better—by "*zebra-line*" relaxation [6, 18]. By zebra-line relaxation we here mean a Gauss-Seidel line relaxation process consisting of two half steps, where in the first half step the even lines and in the second half step the odd lines are relaxed.

In the total-reduction context, a special difficulty arises from the fact that the operator L cannot consistently be approximated by a 5-point difference operator on the rotated grid Ω_H. With respect to this difficulty, in the MGR context, the combination of multigrid with appropriate versions of the "*alternating-reduction* (AR) *method*" [16, 19] (instead of total reduction) is of particular interest. In AR the use of the rotated intermediate grid Ω_H between Ω_h and Ω_{2h} is avoided; instead the AR intermediate grid is characterized by doubling the mesh size in only one direction (x or y). Thus the $(h \to 2h)$ transfer in the xy-AR (or yx-AR, respectively) process is performed by two half steps using the intermediate grid Ω_{2h_x, h_y} (or $\Omega_{h_x, 2h_y}$, resp.).

The MG-AR versions of MGR yield highly efficient algorithms, especially for anisotropic operators. We give results (spectral radii ρ_h^{2h}) for only one algorithm of this type; a systematical study will be contained in [20].

The algorithm [$(h, 2h)$ version] is characterized by the following components (with notation analogous to the description of MGR-CH in Section 4):

[0] ν steps of y-zebra-line relaxation on Ω_h,

[I] defect restriction to Ω_{2h_x, h_y} by the restriction operator

$$\tfrac{1}{4}[1 \quad 2 \quad 1]_h,$$

[II] defect restriction to Ω_{2h} by the restriction operator

$$\frac{1}{4}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}_h ,$$

[III] exact solution on Ω_{2h}, using the discrete operator

$$L_{2h} = \frac{1}{4h^2}\begin{bmatrix} & -1 & \\ -\varepsilon & 2+2\varepsilon & -\varepsilon \\ & -1 & \end{bmatrix}_{2h},$$

[IV] $\left(\Omega_{2h} \to \Omega_{2h_x, h_y}\right)$ transfer using the discrete defect equations with operator

$$L_{2h_x, h_y} := \frac{1}{4h^2}\begin{bmatrix} & -4 & & & \\ -\varepsilon & 0 & 8+2\varepsilon & 0 & -\varepsilon \\ & & -4 & & \end{bmatrix}_h$$

(tridiagonal systems),

[V] $\left(\Omega_{2h_x, h_y} \to \Omega_h\right)$ transfer using the discrete defect equations with L_h (tridiagonal systems).

This algorithm can be analyzed rigorously by means of Fourier analysis as outlined for the $(h, 2h)$ versions of MGR-CH in Section 4. We obtain the results in Table 5 for the spectral radii ρ_h^{2h} with respect to ε and ν ($= 0, 1$).

From these results one immediately recognizes that also for this AR variant of MGR we obtain the bound of $\frac{1}{2}$ for the spectral radius in the case $\nu = 0$. *This bound is independent of* ε. (Note that [IV] and [V] can be interpreted as half steps of zebra-line relaxation, on Ω_{2h_x, h_y} in the x-direction and on Ω_h in the y-direction, respectively. Thus, also in this algorithm, we have an "inherent smoothing"—cf. Remark 2.3—of alternating type.)

In the case $\nu = 1$, of course, the y-direction is distinguished by the y-zebra-line relaxation process. The y-line relaxation is "correct" for the case $\varepsilon \leqslant 1$. Indeed we obtain the well-known value of $0.074 \approx \frac{2}{27}$ if $\varepsilon \to 0$; but even for $\varepsilon \to \infty$ the spectral radii are bounded by $\frac{1}{2}$.

An efficient method which is robust with respect to $\varepsilon \geqslant 1$ *and* $\varepsilon \leqslant 1$ is obtained if *two alternating* steps of zebra-line relaxations are performed in connection with MG-AR, namely, one in the x- and one in the y-direction.

TABLE 5

ρ_h^{2h} FOR THE MGR-AR ALGORITHM[a]

ν \ ε	0.01	0.1	1	10	100
0	0.5	0.5	0.5	0.5	0.5
1	0.074	0.074	0.084	0.25	0.41

[a] Two significant figures; ν steps of y-zebra-line relaxation, $h = \frac{1}{64}$.

Furthermore, such alternating (zebra-) line relaxations are of particular interest if ε is a function of (x, y) which varies in value between $\gg 1$ and $\ll 1$ over the domain Ω.

Finally, we would like to point out that the AR variants of MGR are of particular interest for *3D problems*. A survey of various intermediate grids for 3D problems is contained in [12].

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Furthermore, we also had interesting discussions on the MGR complex with Th. Meis (Cologne) and W. Hackbusch (Bochum). Motivated by the paper of Braess [1] and by our work, Meis has also started to investigate multigrid methods that make use of intermediate grids [11]. Hackbusch has already used checkered relaxation techniques — independently of the MGR — within his MG approach in [8].

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